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On the Control of Linear Multiple Input-Output Systems*

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The control of linear time-invariant systems is one of the most basic problems of modern automatic control theory. Although "optimal controllers" which minimize certain costs associated with control can be determined, in most applications "simple controllers" suffice, and are often more desirable. The criteria by which these simple controllers are designed are closely related to the problem of assigning the eigenvalues of the fundamental matrix (i.e., the poles of the system) to arbitrary but specified locations. This paper presents an approach to the design of such control systems. Our approach does not involve computing complicated canonical forms, as do some previous methods, and at the same time generalizes easily to multi-input-output systems. A simple solution of the problem of designing feedback control systems with a minimum number of dynamic elements is also presented.

I. INTRODUCTION

In recent years there has been a considerable amount of interest in the problem of designing controllers for linear systems. Although most of the theoretical interest has centered around optimal control approaches, it is generally known that in most standard control systems, simple and usually nonoptimal controllers suffice. One of the oldest problems of control theory is that of stabilizing a linear control system by using feedback (see Fig. 1). Although this problem has been solved in the single input-output case by many people, one of the first clear statements was that by D. G. Luenberger.[1] In the case of multiple input-outputs, elegant solutions are of recent origin (see Ref. 2). Almost all of the published solutions resort to canonical forms, and in the multiple input-output case are not convenient to work

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* A talk based on this paper was presented at the Second Assilomar Conference on Circuits and System Theory, 1968.

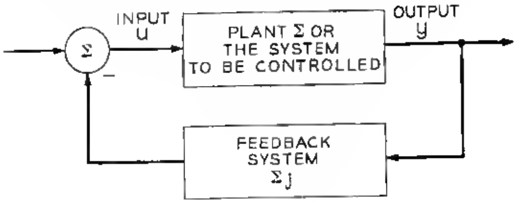

Fig. 1—Model of controller.

with. Also, in almost all cases, since the system is often described in terms of variables that are of direct interest, a transformation to canonical form is inconvenient.

In this memo we present a solution of the problem including the problem of designing controllers of minimal dynamic order (i.e., a controller using the least number of dynamic elements). The present solution does not resort to the use of canonical forms for the design. This approach also helps to systematically exploit the additional freedom that is obtainable due to the multiplicity of the inputs and outputs. In fact there is no previous solution known to the author which solves the problem of designing "minimal order" observers without resorting to complicated canonical forms.

We begin by introducing certain preliminaries and establishing the notation, and then solving the problem of designing controllers and observers in Sections III and IV. In Section V, the problem of designing controllers of low dynamic order is solved.

After this paper was written, the author became aware of the paper by W. M. Wonham.[3] Wonham derives Lemma 1 in the following section. The proof given in Wonham however uses the theory of minimal polynomials, as compared to the proof given in the following section which uses only the concept of linear spaces. Wonham himself has commented in his paper that an abstract proof of his results would be very worthwhile. The author feels that the proof given in the following section is an abstract version. The results of Sections III through V are not to be found in Wonham's paper.

II. LINEAR TIME-INVARIANT SYSTEMS AND THE PROBLEM OF CONTROL

The following definitions contain certain undefined but generally understood concepts such as dynamical system, etc. For a more detailed discussion of these ideas, the reader is referred to Ref. 4.

2.1 *Linear Time-Invariant System*

A linear time-invariant system Σ is a dynamical system governed by the following equation.

$$\dot{x}(t) = Fx(t) + Gu(t), \tag{1}$$

$$y(t) = Hx(t), \tag{2}$$

where $x(t) \in E^n$ is the state of Σ and $u(t) \in E^m$ and $y(t) \in E^p$ are the inputs and outputs of Σ respectively. F, G, and H are $n \times n$, $n \times m$ and $p \times n$ matrices respectively, and are *independent of time* t.

A "system" hereinafter shall denote a linear time-invariant system for brevity.

2.1.1 *Cyclic System*

Σ is cyclic if there exists $x \in E^n$ sucb that the matrix $[x \; Fx \cdots F^{n-1}x]$ is nonsingular.

2.1.2 *Complete Controllability*

Σ is completely controllable if the rank of $[G \; FG \cdots F^{n-1}G]$ is n. See Ref. 4 for details.

2.1.3 *Complete Observability*

Σ is completely observable if the rank of $[H' \; F'H' \cdots F'^{n-1}H]$ is n. Finally Σ is ordinary if

(*i*) Σ is cyclic; and
(*ii*) Σ is completely controllable and observable.

Most systems ordinarily dealt with are cyclic, because, as will be shown in this section, the condition of not being cyclic is caused by having two identical subsystems embedded in one system and yet completely decoupled from each other. Hence it is a singular situation in the sense that whenever a system is completely reachable and completely observable, a slight amount of feedback can make the system cyclic (see Ref. 5).

It is very interesting to note that most theorems to be given in this paper are dependent on a simple and basic property of linear spaces. This property is stated as the following lemma.

Lemma 1: *Let $\mathcal{S}_i$, $i = 1 \cdots n$, be n distinct linear subspaces of a linear space. Let $\mathcal{L}$ be a linear space contained in the set union of $\mathcal{S}_i$,*

$i = 1 \cdots n$. Then

$$\mathcal{L} \subseteq \mathcal{S}_j \text{ for some } j \,\varepsilon\, \{1, 2, \cdots n\}, \tag{3}$$

where $\subseteq$ denotes "contained in".

Proof: The proof will he hased on the principle of finite induction. The lemma is ohviously true for $n = 1$. Now suppose the lemma is true for all $n < n_0$; i.e., suppose that, given $\mathcal{S}_i$, $i = 1, 2, \cdots, n < n_0$, then $\mathcal{L} \subseteq \bigcup_{i \leqq n} \mathcal{S}_i$ implies that $\mathcal{L} \subseteq \mathcal{S}_j$ for some $j \leqq n$. It will then he proved that the lemma is true for $n = n_0$, which will complete the proof of the lemma hy finite induction.

Assume $\mathcal{L}$ is not contained in the set union of any m ($m < n_0$) of the $\mathcal{S}_i$'s, for if it is so contained, then the lemma trivially holds (from previous paragraph). Therefore there exists n_0 vectors x_i such that

$$x_i \,\varepsilon\, \mathcal{L}, \quad x_i \,\varepsilon\, \mathcal{S}_i \quad \text{and} \quad x_i \not\varepsilon\, \mathcal{S}_j \quad \text{if} \quad i \neq j. \tag{4}$$

Consider now any two of these n_0 vectors x_i, x_j and

$$x_i + \alpha x_j, \qquad i \neq j, \qquad \alpha \,\varepsilon\, R. \tag{5}$$

(The set of real numhers).

Since $\mathcal{L}$ is linear, $x_i + \alpha x_j \,\varepsilon\, \mathcal{L} \;\forall\; \alpha$; also, since $\mathcal{L} \subseteq \bigcup_{i \leqq n_0} \mathcal{S}_i$

$$x_i + \alpha x_j \,\varepsilon\, \mathcal{S}_s \quad \text{for some} \quad s(\alpha) \,\varepsilon\, \{1, 2, \cdots, n_0\}. \tag{6}$$

However, since there are only a finite numher of $\mathcal{S}_i$'s while α can assume any values from the uncountahly infinite set R, there exists an "s" such that for at least two distinct values of α, namely α_1 and α_2,

$$x_i + \alpha_1 x_j \qquad \text{and} \qquad x_i + \alpha_2 x_j \,\varepsilon\, \mathcal{S}_s\,.$$

But this implies that

$$(\alpha_1 - \alpha_2) x_j \,\varepsilon\, \mathcal{S}_s \text{ since } \mathcal{S}_s \text{ is linear;} \tag{7}$$

i.e.,

$$x_j \,\varepsilon\, \mathcal{S}_s \text{ since } \alpha_1 \neq \alpha_2 \text{ and } \mathcal{S}_s \text{ are linear.} \tag{8}$$

Therefore $s = j$ hy (4), i.e., $x_i + \alpha_1 x_j \,\varepsilon\, \mathcal{S}_j$, whence $x_i \,\varepsilon\, \mathcal{S}_j$ since $x_j \,\varepsilon\, \mathcal{S}_j$ and $\mathcal{S}_j$ is linear. Once again using (4), $x_i \,\varepsilon\, \mathcal{S}_j \Rightarrow i = j$ which contradicts equation (5). Q.E.D.

Note: As can easily he seen, Lemma 1 does not hold in general for an uncountahle union of linear spaces.

Definition: A square matrix F has simple structure if and only if for

each eigenvalue λ_i, of F there exists one and only one eigenvector e_i. (In other words, F is simple if no two uncoupled Jorden blocks in the canonical form have the same eigenvalue.)

Note: All the eigenvectors are assumed to be normalized such that the first nonzero component is $+1$.

Lemma 2: The statement that the system Σ is cyclic implies that the square matrix F in equation (1) has simple structure.

Proof: Suppose there exists two eigenvectors e_1 and e_2 of F corresponding to the eigenvalue λ. Then $\exists$ two eigenvectors, d_1 and d_2 of the matrix F' corresponding to $\bar{\lambda}$, where F' is the conjugate transpose of F. Let x be any vector in E^n. Then suppose y is the projection of x in $R(d_1, d_2)$, the subspace spanned by d_1 and d_2. Let

$$z \in R(d_1, d_2) \quad \text{such that} \quad (z, y)^* = 0, \quad z \neq 0. \tag{9}$$

Then since $((x - y), z) = 0$ because $z \in R(d_1, d_2)$, it follows that $(x, z) = 0$ by equation (9) and since $z \in R(d_1, d_2)$, $z = \alpha_1 d_1 + \alpha_2 d_2$.

Therefore

$$\begin{aligned}(z, F^r x) &= ((\alpha_1 d_1 + \alpha_2 d_2), F^r x),\\ &= \bar{\lambda}^r((\alpha_1 d_1 + \alpha_2 d_2), x),\\ &= \bar{\lambda}^r(z, x) = 0 \qquad \forall \quad r.\end{aligned}$$

Therefore the proof is complete. Q.E.D.

Definition: A subspace $\mathfrak{L}$ of E^n is an invariant subspace of F if $x \in \mathfrak{L} \Rightarrow Fx \in \mathfrak{L}$. $\mathcal{I}_s(F)$ denotes an s-dimensional invariant subspace of F.

Lemma 3: The statement that Σ is ordinary implies that $\exists$ an $\alpha \in E^m$ and $\beta \in E^p$, such that

$$\rho([F : G\alpha]) = \rho([F' : H'\beta]) = n^\dagger.$$

Proof: Notice that the number of invariant subspaces of F are finite, since the number of one-dimensional invariant subspaces are finite. [This follows from the familiar structure of invariant subspaces, see equation (6).] Suppose $x \in R(G)$, the space spanned by the columns of G and

$$\rho([F : x]) = s(x) < n.$$

* (z, y) is inner product of z and y.

† The matrix $[F : A]$ will in general denote the matrix $[A \; FA \; \cdots \; F^{n-1}A]$, and $\rho([F : A])$ denotes the rank of $[F : A]$.

Therefore $x \in \mathcal{R}(G)$ belongs to some $\mathcal{J}_s(F)$, $s < n$. Therefore, from Lemma 1 $\mathcal{R}(G) \subseteq \mathcal{J}_s(F)$ for some $s < n$, which contradicts $\rho([F:G]) = n$. Therefore $\exists$ an $\alpha \in E^m$, such that $\rho([F : G\alpha]) = n$. Similarly the other case. Q.E.D.

The above lemma shows that $\exists$ a single input-output system corresponding to every ordinary system, such that the controllability and observability of the new system is implied by that of the old system with multiple inputs and outputs. Lemma 4 shows that the weighting vectors α and β could almost be any vector in E^m and E^p respectively.

*Lemma 4: The statement that Σ is ordinary, that $\alpha \in E^m$ and $\beta \in E^p$, implies that $\rho([F : G\alpha]) = \rho([F' : H'\beta]) = n$ almost surely.**

Proof: Notice the determinant of $[F : G\alpha]$ is a polynomial in α, and by Lemma 3 we have shown is nonzero for at least one α. If the distribution of α does not allow nonzero probability to any surface of dimension $< m$ then

$$\text{Probability that } \rho([F : G\alpha]) = n \text{ is } 1. \quad \text{Q.E.D.}$$

The stability of Σ, and the transient response of Σ are generally characterized by the eigenvalues of F, which in turn are given by the characteristic polynomial $\chi(F)$. Hence loosely by dynamics of Σ we mean the characteristic polynomial or the eigenvalues of F.

Given that a system Σ models a "plant" to be controlled the problem that we will consider is that of designing another system Σ_f such that the resultant "closed loop" system has arbitrary dynamics.

III. THE DESIGN OF CONTROLLERS

In order to motivate the nature of the problem in Section IV, we shall first solve the so-called control problem which essentially is a simplified version of the problem postulated in Section II. The H matrix in equation (2) is now assumed to be the "n" dimensional identity denoted by I_n, in other words, the complete state of Σ is available for measurement. In this case, we show that we need only feed back a certain linear function of the state "x" to achieve any given dynamics for the closed loop system.

The problem formally reduces to the following.

Given a plant Σ described by equations (1) and (2) with H in (2)

* α and β are chosen from any joint distribution in E^m and E^p respectively, which does not assign nonzero probability to a surface of dimension less than m and p, respectively.

replaced by I_n, it is required to find an $m \times n$ matrix K referred to in the following as the feedback gain such that the resultant system has the prescribed characteristic polynomial (10)

$$s^n + \sum_{i=1}^{n} \gamma_i s^{n-i}. \tag{10}$$

Let the characteristic polynomial $\chi(F)$ be

$$\chi(F) = s^n + \sum_{i=1}^{n} a_i s^{n-i}. \tag{11}$$

Then from Fig. 2 the problem reduces to finding K such that

$$\chi(F - GK) = s^n + \sum_{i=1}^{n} \gamma_i s^{n-i} \tag{12}$$

since the new differential equation is $\dot{x} = (F - GK)x + Gu$. The solution is contained in the following theorem.

Theorem 1: If a system Σ *is cyclic and completely controllable then with* $\gamma_1, \gamma_2, \cdots, \gamma_n$ *real constants,*

(1)
$$\chi(F - GK) = s^n + \sum_{i=1}^{n} \gamma_i s^{n-i} \tag{13}$$

for any K *of rank one satisfying*

$$\begin{aligned} \gamma_1 &= a_1 + \operatorname{tr}(GK) \\ &\vdots \\ \gamma_n &= a_n + a_{n+1} \operatorname{tr}(GK) + \cdots + \operatorname{tr}(F^{n-1}GK). \end{aligned} \tag{14}$$

(2) *Moreover there exists at least one* K *satisfying equation* (14).

Proof: The new characteristic polynomial with feedback is

$$\chi(F - GK) = \det(sI - F + GK),$$

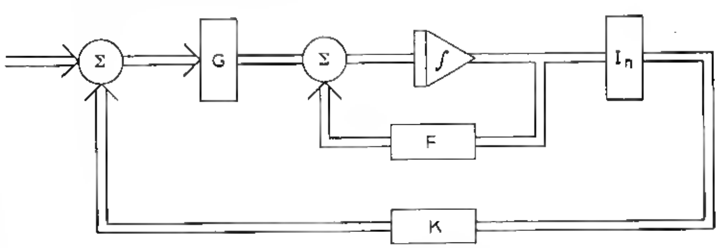

Fig. 2—Feedback control when the state is known.

i.e.,

$$= \det [(sI - F)(I + (sI - F)^{-1}GK)], \tag{15}$$

$$= \chi(F) \det (I + (sI - F)^{-1}GK).$$

Since K is of rank one, it follows that $(sI - F)^{-1}GK$ has rank one; therefore

$$\det (I + (sI - F)^{-1}GK) = 1 + \operatorname{tr} (sI - F)^{-1}GK \tag{16}$$

where tr (A) denotes the sum of the diagonal elements of A. Therefore from equation (15)

$$\chi(F = GK) = \chi(F)[1 + \operatorname{tr} (sI - F)^{-1}GK], \tag{17}$$

i.e.,

$$\chi(F - GK) = \chi(F) + \operatorname{tr} [\chi(F)(sI - F)^{-1}GK] \tag{18}$$

which since

$$(sI - F)^{-1} = \sum_{i=0}^{\infty} F^i s^{-(i+1)}$$

outside an appropriate region in the complex plane (see Ref. 6), becomes

$$\chi(F - GK) = \chi(F) + \operatorname{tr} \left[\chi(F) \sum_{i=0}^{\infty} [F^i s^{-(i+1)} GK] \right] \cdot \tag{19}$$

Now using the Cayley-Hamilton theorem (see Ref. 6), i.e., using the fact that

$$F^n + \sum_{i=1}^{n} a_i F^{n-i} = 0. \tag{20}$$

and equating coefficients of equation (19), we have that the coefficient of s^{n-j} on R. H. S. of equation (19) is (using $a_0 \triangleq \gamma_0 \triangleq 1$)

$$= 0, \qquad j > n; \tag{21}$$

$$\gamma_j = a_j + a_{j-1} \operatorname{tr} GK + \cdots + \operatorname{tr} F^{j-1}GK, \qquad n \geq j \geq 1; \tag{22}$$

$$= 0, \qquad j < 0. \tag{23}$$

Therefore

$$\begin{aligned} \gamma_1 &= a_1 + \operatorname{tr} GK, \\ \gamma_2 &= a_2 + a_1 \operatorname{tr} GK + \operatorname{tr} FGK, \\ &\vdots \\ \gamma_n &= a_n + \cdots + \operatorname{tr} F^{n-1}GK. \end{aligned} \tag{24}$$

This proves that if there exists K of rank 1 such that equation (14) is satisfied, then K satisfies equation (24), and that if K satisfies equation (24), then equation (12) is satisfied.
Let

$$[\gamma_1, \cdots, \gamma_n] \triangleq \gamma', \qquad [a_1, \cdots, a_n] \triangleq a', \tag{25}$$

where $'$ denotes the transpose. We have, rewriting equation (24),

$$\gamma = a + \begin{bmatrix} 1 & 0 & \cdots & 0 \\ a_1 & 1 & \cdots & 0 \\ \vdots & a_1 & \cdots & \cdot \\ a_{n-1} & a_{n-2} & \cdots & a_1 1 \end{bmatrix} \begin{bmatrix} \operatorname{tr} GK \\ \operatorname{tr} FGK \\ \vdots \\ \operatorname{tr} F^{n-1}GK \end{bmatrix}. \tag{26}$$

Let

$$A \triangleq \begin{bmatrix} 1 & 0 & \cdots 0 & 0 \\ a_1 & 1 & \cdots 0 & 0 \\ \vdots & & \vdots & \vdots \\ a_{n-2} & a_{n-3} & \cdots 1 & \cdot \\ a_{n-1} & a_{n-2} & \cdots a_1 & 1 \end{bmatrix}. \tag{27}$$

Notice here that A^{-1} always exists and can be evaluated easily.

$$\begin{bmatrix} \operatorname{tr} GK \\ \operatorname{tr} FGK \\ \vdots \\ \operatorname{tr} F^{n-1}GK \end{bmatrix} = A^{-1}(\gamma - a). \tag{28}$$

Now we assume $K = \alpha k'$ (α and k are $m \times 1$ and $n \times 1$ matrices respectively), such that K is of rank 1 then equation (28) becomes

$$\begin{bmatrix} \operatorname{tr} G\alpha k' \\ \operatorname{tr} FG\alpha k' \\ \vdots \\ \operatorname{tr} F^{n-1}G\alpha k' \end{bmatrix} = A^{-1}(\gamma - a). \tag{29}$$

Since tr $F^iG\alpha k' = \alpha'G'F'^ik$, equation (29) becomes

$$\begin{bmatrix} \alpha'G' \\ \alpha'G'F' \\ \vdots \\ \alpha'G'F'^{n-1} \end{bmatrix} k = A^{-1}(\gamma = a). \tag{30}$$

But from Lemma 4 it is clear that $\rho([F\ G\alpha]) = n$ for almost all α. Therefore it follows that equation (30) has a unique solution for almost any α, and this completes the proof.

Therefore from the proof of the above theorem, it is easy to see how we can find the required gain matrix. Equation (28) is linear in the elements of K. The freedom in the multi input-output case is essentially one of picking α. Almost any solution of equation (28) which has rank 1 will do the job. Notice that restricting K to be of rank 1 also helps to reduce the number of amplifiers to implement the system, for K then can be realized by $(m + n - 1)$ amplifiers instead of (mn).

IV. DESIGN OF OBSERVERS

In Section III we saw how a gain matrix K could be computed for the system Σ with $H = I_n$. However when $H \neq I_n$, the state 'n' of Σ is not directly observable and an observer to estimate the state has to be designed. It will become clear that Theorem 1 gives the solution to this problem also. The solution consists of designing a linear system Σ_0 which is constructed in such a way that its state $\hat{x}$ can easily be observed, and such that the state of Σ_0 tends to the state of Σ as "rapidly as desired." (The meaning will become clear in the following.)

The system Σ_0 will consist of a model of Σ driven by an input which is equal to the sum of the inputs to a weighted error term which is the difference between the state of Σ and that of Σ_0.

Let Σ_0 be defined by

$$\dot{\hat{x}} = F\hat{x} + LH(x - \hat{x}) + Gu. \tag{31}$$

Let the error $\tilde{x} \triangleq x - \hat{x}$. Then equations (1) and (31) imply

$$\dot{\tilde{x}} = F\tilde{x} - LH\tilde{x}. \tag{32}$$

Now we would like $\tilde{x}$ to decrease to zero according to some dynamics in the sense that $\chi(F - LH)$ should be some prescribed polynomial. It is obvious that once again the problem is to find an L such that

$\chi(F - LH)$ is a prescribed polynomial which by Theorem 1 again is easily done by solving

$$\begin{aligned}\gamma_1 &= a_1 + \operatorname{tr} LH,\\ \gamma_2 &= a_2 + a_1 \operatorname{tr} LH + \cdots + \operatorname{tr} LHF,\\ &\vdots\\ \gamma_n &= a_n + a_{n-1} \operatorname{tr} LH + \cdots + \operatorname{tr} HF^{n-1},\end{aligned} \tag{33}$$

where γ_i are the coefficients of the prescribed polynomial. Hence now the complete solution can be stated as follows.

Step 1: By solving equation (33), construct an observer with dynamics such that it is sufficiently fast compared with the plant.

Step 2: By solving equation (26), construct the controller as in Section III to have the required closed loop dynamics.

Step 3: Cascade the observer and the controller as in Fig. 3.

We can show that the characteristic polynomial of the entire closed loop system of Fig. 3 is actually $\chi(F - GK)\chi(F - LH)$.

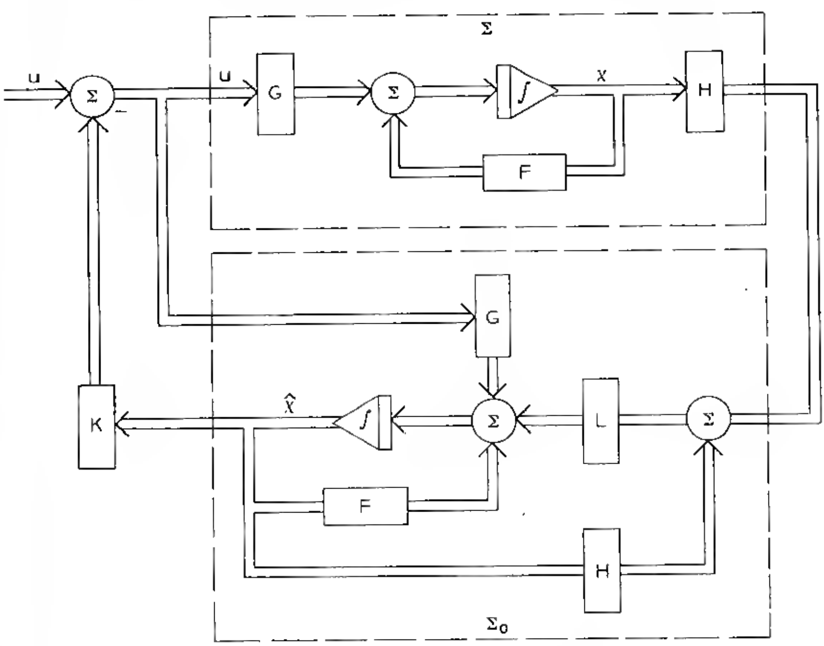

Fig. 3—The structure of a complete "controller."

The differential equation for the complete system of Fig. 3 is as follows.

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} F & -GK \\ LH & F - GK + LH \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} G \\ G \end{bmatrix} u.$$

Since the characteristic equation is unaltered by a nonsingular transformation of the state $[x', \hat{x}']'$, consider the transformation T denoted by

$$\begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}.$$

Obviously T is nonsingular and

$$T \begin{bmatrix} F & -GK \\ LH & F - GK + LH \end{bmatrix} T^{-1} = \begin{bmatrix} F - GK & GK \\ 0 & F - LH \end{bmatrix}$$

whose characteristic polynomial is clearly

$$\chi(F - GK)\chi(F - LH).$$

V. DESIGN OF OBSERVERS OF MINIMAL ORDER

In Section IV, the design of observers was discussed and it was seen that the observer is a system of the same order as Σ, namely n. If it is assumed that H has full rank (this assumption is obviously no loss of generality), it is clear that the knowledge of Hx gives measurements on part of the state immediately; for

$$y = Hx \tag{34}$$

gives the projection of x on the row space of H. Hence it seems that one should be able to find the other component in the complement of the row space of H by making use of a dynamical system of order $(n - p)$ $(H\colon p \times n)$. This was first proved by Luenberger in 1964 in the single input/output case[1] and was proved in the multiple input/output case by the same author in 1966.[2]

In the following section, a simple proof of this proposition is given and a different method for constructing the observer is derived.

Let the given system be, as before:

$$\begin{aligned} \dot{x} &= Fx + Gu, \\ y &= Hx. \end{aligned} \tag{35}$$

Let H be of full rank. It can be assumed that H is of the form

$$H = [I_p, 0], \qquad p < n; \tag{36}$$

since a nonsingular transformation of the state of equation (35) can always be found such that equation (36) holds. Hence, if

$$x = \begin{array}{c} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{array}\begin{array}{l} p \\ n-p \end{array}, \tag{37}$$

and

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}\begin{array}{l} p \\ n-p \end{array}, \tag{38}$$

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}\begin{array}{l} p \\ n-p. \end{array} \tag{39}$$

Then

$$\begin{aligned} \dot{x}_1 &= F_{11}x_1 + F_{12}x_2 + G_1u, \\ \dot{x}_2 &= F_{21}x_1 + F_{22}x_2 + G_2u. \end{aligned} \tag{40}$$

Now the design procedure will be described in the form of two theorems for clarity.

Theorem 2: If (H, F) is completely observable, then $[F_{12}, F_{22}]$ is completely observable.

Proof: The assertion of the theorem is in some sense intuitively clear, since $y = x_1$ does not give any direct information about x_2. The only information about x_2 is obtained from equation (40). That is,

$$F_{12}x_2 = \dot{x}_1 - F_{11}x_1 - G_1u, \tag{41}$$

which implies that F_{12}, F_{22} should be completely observable in order that (H, F) be completely observable.

The proof of the theorem is immediate, since (H, F) completely observable implies that the rank

$$\rho[(H', F'H', \cdots, F'^{n-1}H')] = n,$$

i.e., using the partition of F indicated in equation (38),

$$\rho\left[\begin{bmatrix} I & 0 \\ F_{11} & F_{12} \\ F_{11}^2 + F_{12}F_{21} & F_{11}F_{12} + F_{12}F_{22} \\ \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}\right] = n. \tag{42}$$

[In the following a row of a partitioned matrix will mean the "block" row; for example, the first row of the matrix in equation (42) is $[I\ 0]$, the second row is $[F_{11}\ F_{12}]$, etc.]

The rank of the matrix in equation (42) is unaltered by adding to any row linear combination of other rows. Hence

$$\rho\left(\begin{bmatrix} I & 0 \\ \cdots & F_{12} \\ \cdots & F_{12}F_{22} \\ \cdots & \cdots \\ \cdots & F_{12}F_{22}^{n-1} \end{bmatrix}\right) = n. \tag{43}$$

The third row of the matrix in equation (43) is the third row of the old matrix $- F_{11}^2$ (first row) $- F_{11}$ (second row), etc. Equation (43) implies that, irrespective of what the first column of the matrix in equation (43) is,

$$\rho\left(\begin{bmatrix} 0 \\ F_{12} \\ \vdots \\ F_{12}F_{22}^{n-1} \end{bmatrix}\right) = (n - p), \tag{44}$$

which by the Cayley–Hamilton theorem implies that one need only include terms up to $F_{12}F_{22}^{n-p-1}$ which gives

$$\rho([F'_{12}\ , F'_{22}F'_{12}\ , \qquad , F'^{n-p-1}_{22}F'_{12}]) = (n - p). \tag{45}$$

Theorem 3: There exists an "observer" of dimension $(n - p)$ for Σ of Theorem 2.

Proof: Consider the "partitioned" system presented in Theorem 3.

From Section IV it is clear that since $(F_{12}\ , F_{22})$ is completely observable we can find an L such that $(F_{22} - LF_{12})$ has arbitrary eigenvalues. Hence if $\bar{x}_2$ is defined by

$$\dot{\bar{x}}_2 = F_{22}\bar{x}_2 + LF_{12}(x_2 - \bar{x}_2) + G_2u + F_{21}x_1\ , \tag{46}$$

then $\tilde{x} = x_2 - \bar{x}_2$ implies

$$\dot{\tilde{x}} = (\dot{x}_2 - \dot{\bar{x}}_2) = (F_{22} - LF_{12})\tilde{x}_2\ . \tag{47}$$

Therefore by choosing L appropriately, we can make $\tilde{x}_2 \rightarrow 0$ as fast

as we want. The only problem is observing $(x_2 - \bar{x}_2)$ in equation (46), that can be solved as follows. Using equation (40), equation (46) reduces to

$$\dot{\bar{x}}_2 = (F_{22} - LF_{12})\bar{x}_2 - LF_{11}x_1 - LG_1u + G_2u + F_{21}x_1 + L\dot{x}_1 . \quad (48)$$

All the terms on the R. H. S. of equation (48) except $L\dot{x}_1$ can be observed. In order to eliminate the need for getting $L\dot{x}_1$ we replace it by $(F_{22} - LF_{12})Lx_1$, i.e., if

$$\dot{\hat{x}}_2 = (F_{22} - LF_{12})\hat{x}_2 - LF_{11}x_1 - LG_1u + G_2u + F_{21}x_1 + (F_{22} - LF_{12})Lx_1 . \quad (49)$$

Then it can be shown by integration by parts that

$$\bar{x}_2 = \hat{x}_2 + \exp(F_{22} - LF_{12})t \cdot(\bar{x}_2(0) + LF_{11}x_1(0) + LG_1u(0) - G_2u(0)) + Lx_1(t). \quad (50)$$

Therefore by appropriately choosing initial conditions for the system described by equation (49) we can make

$$\bar{x}_2 = \hat{x}_2 + Lx_1(t) \quad (51)$$

hence the proof is complete. Note that even if the initial conditions for equation (41) were not set, the error term, namely,

$$\bar{x}_1 - \hat{x}_2 - Lx_1(t) \to 0 \quad \text{as fast as} \quad \exp(F_{22} - LF_{12})t.$$

The proof of the theorem, though complete above, is easier to see in the form of figures.

Equation (48) says that the observer is of the form presented in Fig. 4.

Theorem 3 implies that Fig. 4 is equivalent to Fig. 5, namely, the input $\dot{x}_1$ shifted over to the right of $(n - p)$ dimensional integrator.

From the above theorems the following method evolves for the construction of minimal-order observers.

Step 1: Construct by the method of §3 an L such that $F_{22} - LF_{12}$ is stable and has the required dynamics.

Step 2: Use the output x_1 in the configuration presented in Fig. 5 to get $[x_1', \hat{x}_2']$ which gives the required estimate.

VI. EXAMPLE

In order to illustrate the methods presented in this paper, we solve a simple example of a control problem.

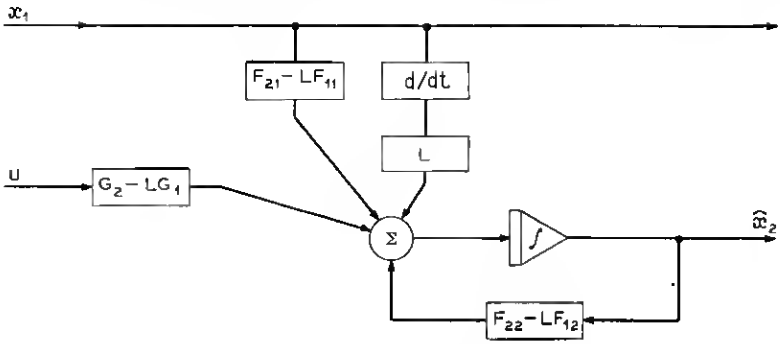

Fig. 4—Minimal order observer with rate information.

The problem is to stabilize the control system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \tag{52}$$

It is readily seen that the characteristic polynomial of this system is

$$s^3 - 3s^2 + 3s - 1, \quad \text{or} \quad (s - 1)^3, \tag{53}$$

which shows that the system is unstable.

It is desired to design a controller which observes the output y and computes a linear feedback law such that the plant (52) behaves as a

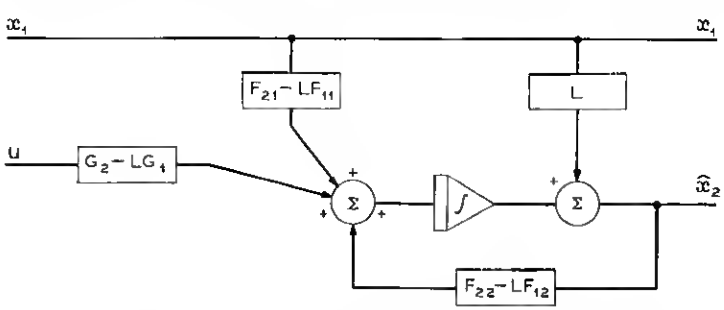

Fig. 5—Minimal order observer without rate information.

system with characteristic polynomial

$$(s + 1)^3 = 0. \tag{54}$$

Proceeding as in Sections III and IV, suppose (x_1, x_2, x_3) were available; then the feedback control gain matrix say

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \end{bmatrix} \triangleq K \tag{55}$$

can be computed from equation (24) which gives

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 1 \\ 9 & 4 & 1 & 2 & 2 & 1 \\ 10 & 6 & 1 & 10 & 4 & 1 \end{bmatrix} \begin{bmatrix} k_{11} \\ k_{12} \\ k_{13} \\ k_{21} \\ k_{22} \\ k_{23} \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \\ 38 \end{bmatrix}. \tag{56}$$

Almost any solution (56), subject to the condition that rank of $K = 1$, will do. For the purpose of this discussion a particular solution, namely

$$\begin{bmatrix} 2/3 & 4/3 & 4/3 \\ 2/3 & 4/3 & 4/3 \end{bmatrix} \tag{57}$$

will be considered.

Now since x_3 is not actually available, an estimate will be computed and the error will be required to diminish as fast as $\exp(-2t)$, at the same time using only one integrator in the feedback loop. From equation (52) it follows that

$$\dot{x}_3 = x_3 + u_1 + u_2 , \tag{58}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} x_3 + \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{matrix} u_2 \\ u_2 \end{matrix}. \tag{59}$$

The expressions in equations (58) and (59) can be compared to the more general treatment of Section III by noting that

$$F_{11} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \qquad F_{12} = \begin{bmatrix} 2 \\ 2 \end{bmatrix};$$
$$F_{22} = [1], \qquad F_{21} = [0 \quad 0]. \tag{60}$$

Then

$$F_{22} - LF_{12} = 1 - [L_1 \quad L_2]\begin{bmatrix} 2 \\ 2 \end{bmatrix} = -2. \tag{61}$$

One solution for L is

$$L_1 = L_2 = \tfrac{3}{4}. \tag{62}$$

The analogue of equation (23) is

$$\dot{\hat{x}}_3 = \hat{x}_3 + (3/2 \quad 3/2)\begin{bmatrix} 2 \\ 2 \end{bmatrix}(x_3 - \hat{x}_3) + u_1 + u_2 \ ; \tag{63}$$

that is, the solution of equation (46) obviously tends to x_3 as fast as $\exp(-2t)$ for from equation (52)

$$\frac{d}{dt}(x_3 - \hat{x}_3) = -2(x_3 - \hat{x}_3). \tag{64}$$

In order to construct the observer, in equation (63), x_3 has to be replaced by x_1, x_2 which are directly measurable. Proceeding just as in §3, equation (46) becomes

$$\dot{\hat{x}}_3 = -2\hat{x}_3 + [3/2 \quad 3/2]\left\{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right\} + u_1 + u \ . \tag{65}$$

Therefore

$$\dot{\hat{x}} = -2\hat{x}_3 - 3/2\, x_1 - 6x_2 - 9/2\, u_1 + 3/2(\dot{x}_1 + \dot{x}_2) + u_1 + u_2 \ . \tag{66}$$

The only quantity that is not available on the right-hand side of equation (66) is $(\dot{x}_1 + \dot{x}_2)$, but appealing to Theorem 4 it follows that if

$$\dot{\bar{x}}_3 = -2\bar{x}_3 - 3/2\, x_1 - 6x_2 - 9/2\, u_1 - 3(x_1 + x_2) + u_1 + u_2 \tag{67}$$

$$\hat{x}_3 \rightarrow \bar{x}_3 + 3/2(x_1 + x_2) \tag{68}$$

and the error tends to zero as $\exp(-2t)$ [it can be made zero by setting the initial condition on the integrator simulating equation (62) to $\hat{x}_3(0) - \frac{3}{2}(x_1(0) + x_2(0))$].

Hence $\bar{x}_3 + \frac{3}{2}(x_1 + x_2)$ which can also be gotten as in Fig. 5 tends to x_3 as $\exp(-2t)$. Therefore the controller design is complete. The analog computer realization is shown in Fig. 6.

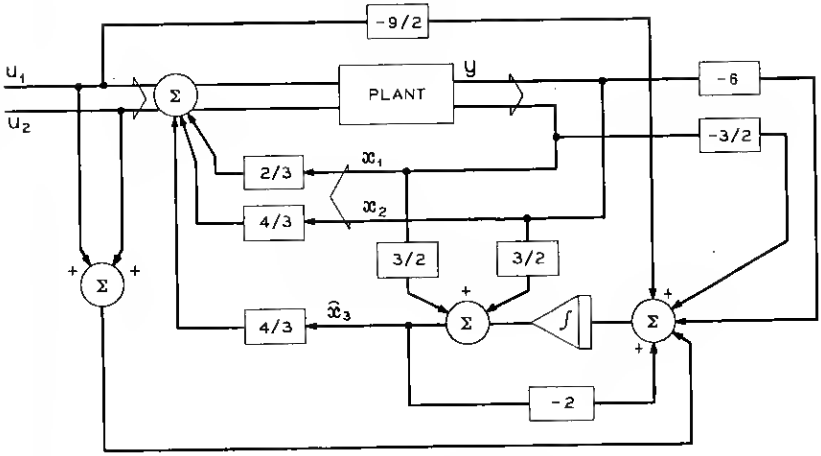

Fig. 6—Example of a controlled system.

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